**About the accuracy of the numerical solution of the energy equation in fluid film lubrication**

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# Introduction

The flow of thin lubricant films in journal and thrust bearings is most often described by governed Reynolds equation of lubrication coupled with the energy transport equation. The numerical solution of these two-coupled equations is a problem solved since many decades. However, it requires a computational effort that can render transient analyses very time consuming. It then becomes important to have efficient solvers and coupling strategy.

If film rupture and reformation (traditionally designed as “cavitation”) are absent and if the flow regime is laminar and isothermal, then Reynolds equation is an elliptic linear differential equation. A direct solver can be used after its discretization in the thin film plane.

The energy equation contains convective transport terms, conductive transport across the thin film and source terms arising from its coupling with the Reynolds equation. Its character is therefore not entirely elliptic and its solver is different from the one used for Reynolds equation.

Moreover, the energy equation must be discretized also across the thin fluid film and the number of discretization points in this direction must be large enough for capturing the wall temperature gradients. The computational effort for the energy equation is then considerably higher than for Reynolds equation. Developing an accurate and efficient solver for the energy equation is then an important step in solving non-isothermal lubrication problems. The task is not simple because there is no analytic solution of the complete energy equation to be used for validations. For example, if the analytic solution of the laminar and isothermal thin film flow in a one dimensional (1D) slider can be used for validating a Reynolds equation solver, no similar solution exists for the energy equation. For validation, a numerical solver for the energy equation will have to be tested with different wall boundary conditions (isothermal or adiabatic) and the numerical results must be checked for various grid densities.

As mentioned, the energy equation must be discretized also across the film thickness. This renders its solution time consuming and the normal practice is to increase the number of discretization cells across the thin film until obtaining grid independent numerical solutions. In natural discretization methods (NDM) the variation of the temperature between two cells is linear or quadratic. This leads to an important number of discretization cells across the thin film. An efficient approach for solving the energy equation makes use of the Legendre polynomials for describing the temperature variation across the thin film. The coefficients of these approximations are obtained by a collocation method applied at the roots of the highest used Legendre polynomial. These roots are the Lobatto points and the method is known as the “Lobatto points collocation method” (LPCM). The method is known since decades but was used only on an ad hoc basis.

The present work presents a systematic comparison between the natural discretization method (NDM) of the energy equation and its LPCM approximation. In order to decouple Reynolds and energy equation, the viscosity is supposed constant and the flow laminar. A one dimensional (1D) inclined slider is used for numerical tests. The Reynolds equation has then an analytic solution and the analysis is entirely focused on the energy equation.

The results for the 1D slider compare the number of points needed by the NDM and by the LPCM for obtaining grid independent results. Both situations, with imposed wall temperatures and wall temperature gradients are analyzed. When wall temperatures are imposed, then the wall temperature gradient is used for estimating the accuracy of the solution and reciprocal. The results show how the NDM is excessively time consuming when high accuracy is needed and the net economy of computational time brought by the LPCM approach.

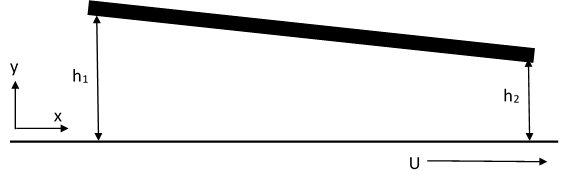
The same analyses is then carried on for a circular journal bearing with a circumferential, central, feeding groove. For evaluation purposes, the film rupture/reformation is dealt with by using the two approaches previously mentioned. The a free boundary formulation of the incompressible Reynolds equation was solved by using an efficient solver based on Fischer-Burmeister form, Newton algorithm and Shur’s complement. The artificial compressibility approach was combined with a regularization technique for avoiding discontinuities and the discretized equation were solved with a Newton algorithm.

The two approaches show comparable results in terms of robustness and computational effort. The LPCM approach was again largely superior to the NDM solution in terms of computational time.

These results show that efficient numerical solutions of the coupled Reynolds and energy equations are possible if a good approximation of the temperature variation across the film is used together with robust film rupture/reformation algorithms.

The proposed test case is an infinite width inclined slider whose length is. It presents a linearly decreasing film thickness varying from 182.88 to 91.44. The upper wall is stationary while the lower wall has a constant velocity 31.946. The density of the lubricant is 800 kg/m3, the specific heat capacity and the thermal conductivity. In order to decouple the energy equation with the Reynolds equation, the constant dynamic viscosity is.

This 1D inclined slide is first described in [1] for the resolution of the Reynolds equation coupled with the energy equation. In difference from the purpose in [1], the current work focus on the resolution of decouple energy equation using analytical solution of Reynolds equation. This gives accurate numerical solution of the energy equation without the influence of the coupling system. Thus, the obtained results could be used for validation of energy equation resolution before coupling with Reynolds equation.



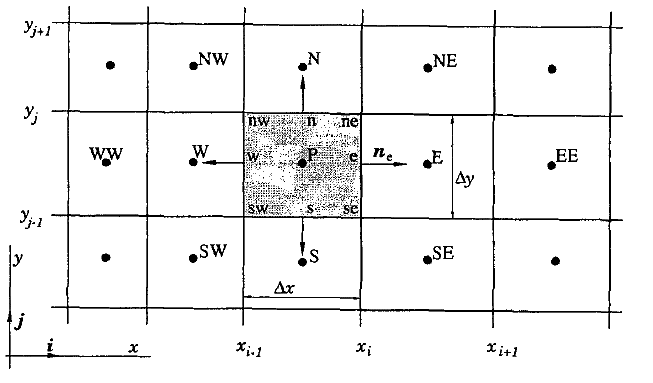
**Figure 1 THE 1D slider**

The data used for the 1D slider are the following:

* Density
* Specific heat capacity
* Thermal conductivity
* Dynamic viscosity
* Lower wall operating speed
* Inlet thickness
* Outlet thickness
* Slide length
* Ambient temperature
* Ambient pressure

# The numerical solution of the energy equation based on the natural discretization method

In the natural discretization numerical approach, the energy equation is solved with the finite volume method. The energy equation (Eq.1) is integrated over a 2D computational cell, which is consists of four plane faces, denoted by lower-case letters corresponding to their direction (e, w, n, s) with respect to the central node P (Figure 2).



**Figure 2 : A typical 2D computational cell and the notation used for a 2D grid**

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In order to build 2D quadrilateral cell for the computational domain, the following coordinate transformation is used:

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Following the coordinate transformation, the energy equation (Eq.1) becomes:

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An upwind discretization technique is used for the convective terms when integrating to avoid numerical instability. For example in east face of control volume, the temperature is up-winded based on the fluid flow direction. Mathematically, this can be express in Eq.4.

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This yields the discretization form of Eq.3:

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| *Where:*  *; ;*  *; ;* |  |

For decoupling the Reynolds and the energy equation the viscosity is considered constant and the flow regime is laminar. The expression of velocity u before coordinate transformation is then

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The velocity component across the thin film, v, is deduced by integrating the continuity equation (Eq.7):

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After integrating it along the film thickness in the computational domain with the boundaries given as and , we get

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Grids with 80 cells in the X direction and a different number of cells across the film thickness were used. The number of equidistant grid cells across the film thickness was Ny = 10, 20, 40, 50, 80, 100,160. In a first test, the temperatures were imposed on the lower and upper wall and the dimensionless wall temperature gradient were monitored.[[1]](#footnote-1) This dimensionless wall temperature gradient is obtained by

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The results are depicted in Figure 3 and Figure 4 and show that the curves become superposed starting with Ny=80 cells. An estimation of the accuracy is obtained by using the relative error (defined in Eq.10) between successive solutions and the wall temperature gradient obtained with the finest grid (Ny=160).

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The variation of these errors with the number of grid cells across the film is depicted in Figure 5. For Ny = 80 the dimensionless error is less than 0.03% and the numerical solution is of acceptable accuracy. The computational time is depicted in Figure 6. The calculation case with Ny = 80 grid cells requires **1,844 sec**. This case will be considered as a reference for the following comparisons.

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| **Figure 3 at the lower wall obtained by NDE** |
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| **Figure 4 at the upper wall obtained by NDE** |

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| **Figure 5 relative error** | **Figure 6 computationnel time** |

# The numerical solution of the energy equation based on Lobatto points collocation method

The Lobatto Point Collocation Method (LPCM) is based on Legendre polynomials used for approximating the variable temperature across the film thickness. Because Legendre polynomials are defined on, the following coordinate transformation is used:

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Following the coordinate transformation, the energy equation (Eq.1) becomes:

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For an incompressible lubricant with constant viscosity, only the variable temperature is approximated across the film thickness (Eq.13).

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| *Where:* |  |

This description holds for every point on the surface of the slider. The coefficients of the Legendre polynomials expansion can be obtained using different methods but it is accepted that the most reliable is the collocation method using the Lobatto points. The Lobatto points are the roots of the derivative of the highest degree of the Legendre polynomial (i.e. the roots of) in Eq.13.

Compared to the NDM which computes directly the temperature by solving the energy equation discretized over the film thickness, the LPCM calculates the polynomial coefficients of temperature. For a given position in x direction, the variable temperature in Eq.12 is replaced by its decomposition (i.e. Eq.13) for each internal Lobatto across the fluid film leading to a partial differential equations with the N+1 unknown. The boundary conditions are applied at the two boundary points and. For example in 1D slider case, the Dirichlet boundary condition (wall temperature ) applied to the upper wall and lower wall can be implemented by setting:

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In all, a system of N+1 equations for the N+1 unknown is obtained.

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| **Figure 7 at the lower wall obtained by LPCM** |
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| **Figure 8 at the upper wall obtained by LPCM** |

The computational effort is thus reduced because only a limited expansion of Legendre polynomials is sufficient for obtaining grid independent results.

Figure 7 and Figure 8 depict the dimensionless wall temperature gradient obtained with an increasing number of Legendre polynomials. The results used for comparison are given by the normal discretization method with Ny = 80 grid cells. The relative error between the reference results and the wall temperature gradient obtained with the LPCM is given by Eq.15 .

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| *for LPCM*  *or for NDM* |  |

Where the wall temperature gradient is calculated by Eq.16 :

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The relative errors are depicted in Figure 9 and show that the approximation of the temperature variation with Legendre polynomials of degree 8 yield grid independent solution. The computational effort is depicted in Figure 10. Compared with the reference solution obtained with the normal discretization method, the computational time is divided by three.

**\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ partie à reprendre par Mihai**

Le résultat du calcul montre qu’à partir de 8 points Lobbato, les courbes du flux thermique aux parois ont tendance de bien suivre la forme de cela de référence. Ceci est notamment constaté sur le flux à la paroi supérieure. Cependant, une différence non-négligeable est remarquée sur le flux à la paroi inférieure (Figure 7) et la convergence de l’erreur est vers 6% (Figure 9). Cette erreur relative élevée est à cause de l’inexactitude de résultat de référence. Si le résultat de référence est celui obtenu avec 160 volumes dans la direction verticale, l’erreur relative diminue vers 2%. Malgré l’erreur relative non-négligeable, la méthode d’approximation Lobbato est ainsi validée.

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| **Figure 9 relative error Lobatto** | **Figure 10 computational time** |
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| **Figure 11 results obtained by LPCM N=10 (he/hs=2)** | **Figure 12 results obtained by NDM (he/hs=2)** |

En outre, la méthode Lobbato est plus rapide en termes de temps de calcul pour une seule résolution du problème. Avec 10 points Lobbato imposé, une seule résolution pourrait gagner 70% du temps de calcul. De plus, lors que le couplage entre l’équation de Reynolds et l’équation de l’énergie est considéré, la résolution du système couplé par l’itération est nécessaire, ce qui demande plusieurs fois de résolution de l’équation de l’énergie pour converger vers un résultat physique. L’avantage de la méthode d’approximation Lobbato sera plus remarquable.

# Comparison of numerical results obtained by NDM and LPCM

Le calcul est initialement fait avec un rapport d’épaisseur du film he/hs=2. Cependant, afin de tester la performance du code de calcul, les calculs avec deux configurations géométriques plus sévères (he/hs=4 et he/hs=8) sont effectués. Puisque la R.C. est plus précis avec plus de points, le résultat obtenu par R.C. avec Ny=160 volume de contrôle dans la direction Y et Nx=80 est figé comme une référence pour valider les résultats obtenus par la méthode d’approximation Lobbato.

1. different geometrical configuration of 1D slider

* Rapport d’épaisseur du film he/hs=4

Le problème est traité avec plusieurs points de Lobbato et les résultats sont présentés ci-dessous :

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| **Figure 13 at the upper wall obtained by LPCM (he/hs=4)** | |
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| **Figure 14 at the lower wall obtained by LPCM (he/hs=4)** | |
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| **Figure 15 dimensionless error Lobatto (he/hs=4)** | **Figure 16 computational time (he/hs=4)** | |
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| **Figure 17 results obtained by LPCM N=10 (he/hs=4)** | **Figure 18 results obtained by NDM (he/hs=4)** | |

Selon le **Figure 13** ***Figure 14*** ***Figure 15*** ***Figure 16*** , on peut constater que la convergence vers le résultat de référence est assurée quand le nombre des points de Lobbato dépasse 10. La méthode d’approximation de Lobbato est plus avantageuse quand des résultats précis sont demandés (maillage plus fin), le temps de calcul peut largement être réduit. Dans ce cas, le temps de calcul par la méthode Lobbato pourrait aller 10 fois moins que la résolution complète pour une seule résolution.

* Rapport d’épaisseur du film he/hs=8

Le même calcul est aussi effectué sur le rapport he/hs=8.

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| **Figure 19 at the upper wall obtained by LPCM (he/hs=8)** | |
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| **Figure 20 at the lower wall obtained by LPCM (he/hs=8)** | |
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| **Figure 21 dimensionless error Lobatto (he/hs=8)** | **Figure 22 computational time (he/hs=8)** | |
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| **Figure 23 results obtained by LPCM N=10 (he/hs=8)** | **Figure 24 results obtained by NDM (he/hs=8)** | |

Selon les **Figure 19 *Figure 20* *Figure 21* *Figure 22***, on peut constater qu’avec une configuration géométrique extrême, il demande plus des points Lobatto pour approximer correctement la distribution de la température. La courbe de l’erreur relative devient stable et se converge à partir de 14 points de Lobatto autours autour de 2.5%. Les autres remarques sont identiques comme dans les parties précédentes.

1. different thermal boundary conditions applied to 1D slider

Afin de faciliter l’implantation dans le nouveau code de palier prochainement, Le code permet de choisir différents conditions limites. Dans le cas du patin incliné, trois conditions limites de la température peuvent être imposées séparément sur la paroi inférieur, la paroi supérieur et à l’entrée du patin.

Pour tester la prise en compte de ces conditions limites, deux calculs avec des conditions limites différente pour un cas he/hs=4 sont effectué. Les résultats des calculs effectué avec 10 points de Lobbato sont comparés avec le résultat de référence obtenu par R.C. avec 80 volumes de contrôle dans la direction X et 160 volumes de contrôle dans la direction verticale.

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| * Boundary condition 1 : T\_sup=30 °C dT/dy\_inf=0 T\_gau=20°C | |
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| **Figure 25 or at the upper wall (BC1)** | |
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| **Figure 26 temperature at the lower wall (BC1)** | |
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| **Figure 27 results obtained by LPCM N=10 (BC1)** | **Figure 28 results obtained by NDM (BC1)** | |

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| Boundary condition 2 : T\_sup=30°C T\_inf=20°C T\_gau=20°C | |
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| **Figure 29 or at the upper wall obtained by LPCM (B.C.2)** | |
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| **Figure 30 or at the lower wall (B.C.2)** | |
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| **Figure 31 results obtained by LPCM N=10 (B.C.2)** | **Figure 32 results obtained by NDM (B.C.2)** | |

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# Results for the energy equation coupled with Reynolds equation

The further study has applied the NDM and the LPCM to the previous 1D slider test case with variable viscosity. With the published numerical data in the literature [1], the validation of numerical solution for the energy equation coupled with Reynolds equation become possible. This step complete the decoupled energy equation validation described in the previous paragraphs to ensure a correct resolution of energy equation for non-isothermal lubrication problems.

To complete the viscosity law in the mentioned 1D slider case, the temperature dependent viscosity follows an exponentially decaying law. The computational domain is discretized using 30 cells in the main flow direction and 10 Lobatto points across the film thickness. Figure 33 depicts the pressure distribution along the main fluid flow direction. The pressure predicted by the LPCM shows good agreement with the published results obtained by Elrod H.G . Figure 34 presents the variation of the outlet temperature across the film thickness. Again, the predictions of the current model show good agreement.

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| ***Figure 33: Pressure distribution*** | ***Figure 34: Outlet temperature distribution across the fluid film*** |

This 1D inclined slider is also used to compare the efficiency of the LPCM compared to the NDM. Several tests are performed with the NDM method in order to obtain a grid independent result which could serve as a reference. These tests use seven grids refined in y direction (10, 20, 40, 60, 80, 100, and 120 equally spaced control volumes) while a constant number of 30 control volumes is used in the x direction. The relative error between two successive grids in terms of wall temperature gradients is defined as:

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| where the subscript K indicates the grid refinement level in the y direction |  |

Figure 35 shows that a minimum number of 40 control in y direction are necessary to reach a satisfactory mesh-independent solution. The computational time is depicted in Figure 36. Thus, the solution obtained by NDM with 40 equally spaced CVs over the film thickness is considered as the reference for this infinite slider test case.

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| ***Figure 35: Relative error of the NDM method for different grid refinements*** | ***Figure 36: Computational time of NDM for different numbers of CVs over the film thickness*** |

The LPCM results obtained with different numbers of Lobatto points are compared with the reference NDM numerical solution. The relative error between the reference temperature gradients at the walls and the ones predicted by using N Lobatto points is defined:

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In Figure 37, the relative error drops rapidly and remains below 2% starting with 11 Lobatto points. Figure 38 shows that the computational time for the LPCM does not exceed 2 seconds while the reference method takes about 18 seconds. The economy of computational time brought by the LPCM in coupled energy equation with Reynolds equation is more evident than in decoupled case.

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| **Figure 37: Relative error between LPCM and the reference NDM results for different numbers of Lobbato points** | **Figure 38: Computational time in function of number of Lobatto points compared to the reference NDM results.** |

The another validation case for the thermal Reynolds equation is a recent experimental result published by Giraudeau et al. [7] in 2016 for a two-lobe journal bearing with an axial supply groove. The tested bearing length is 68.4 mm and its diameter is 100 mm. The radial assembly clearance is 68 while the radial bearing clearance is 143. The bearing is lubricated by an ISO VG 46 oil supplied at a constant pressure of 0.17 MPa and a temperature of 43°C. The following oil characteristics are used for the calculations:, and. The viscosity of the oil is at and at. The variation is described by an exponentially decaying law.

Only the results for the lower, loaded lobe and for the operating condition with significant thermal effects are shown here. The shaft is considered to have a constant temperature estimated from experiments while adiabatic wall conditions are imposed on the bushing. The computational domain is discretized using 3216 control volumes in circumferential and axial directions while 11 Lobatto points are used to describe the temperature variation across the fluid film.

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| **Figure 39: Comparison of measured pressures and current THD solver (3500 rpm, 6kN load)** | **Figure 40: Comparisons of measured temperatures and current THD solver (3500 rpm, 6kN load)** |

Figure 39 and Figure 40 depict the pressure and the temperature variations in the circumferential direction. The predicted pressures show good agreement with the measurements. The predicted temperature shows a reasonable agreement with the measurements and the quality of the prediction could be improved if the thermo-deformation of the bushing is considered and the thermal boundary conditions for the energy equation are refined.

# Conclusion (à faire)

La méthode d’approximation Lobbato est validée en comparant avec la résolution compète. L’intérêt d’utiliser cette méthode d’approximation Lobbato est justifié. Pour une seule résolution, la méthode Lobbato peut aller 10 fois plus rapide par rapport la méthode classique avec un maillage 81\*161. En plus, cette méthode d’approximation devient plus avantageuse lors la résolution de l’équation de l’énergie pour plusieurs fois après couplage avec l’équation de Reynolds.

Cependant, quand la configuration géométrique devient extrême, il nécessite de plus de points Lobbato pour approximer la distribution de la température. En outre, le choix des conditions limite de type Dericlet et Newmenn est aussi implanté dans le script de calcul en se basant sur la méthode Lobbato.

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1. This is equivalent to monitoring the wall heat fluxes. [↑](#footnote-ref-1)